Inverse Electromagnetic Problems for Spherical Scatterers

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***Abstract* – In this Master's Thesis, the inverse electromagnetic scattering problem of determining the geometrical and physical characteristics of a layered sphere is analytically investigated. Efficient algorithms in the low-frequency region are analyzed. These algorithms utilize essentially the information encoded in the scattered far-field due to primary dipoles lying at different locations inside or outside the scatterer.**

Keywords: Direct and inverse scattering problems; Electromagnetic waves; Spherical media

I. INTRODUCTION

Scattering theory usually investigates the interactions of an incident wave with a bounded three-dimensional obstacle. Collections of results for scattering by canonical shapes are included in the classic book by Bowman, Senior, and Uslenghi (1969). In the low-frequency region, acoustic, electromagnetic and elastic scattering problems are analysed systematically in the book by Dassios and Kleinman (2000).

A normalized spherical incident field due to an acoustic point source lying in the exterior of the scatterer was introduced in (Dassios & Kamvyssas, 1995), where low-frequency far-field results in the case of a non-penetrable spherical scatterer were obtained and inverse scattering problems were investigated. The normalization was mainly utilized to retrieve the respective results due to plane incident waves. Low-frequency direct and inverse problems for acoustically penetrable spheres were treated in (Dassios, Hadjinicolaou & Kamvyssas, 1999). Near-field inverse scattering problems for small spheres were investigated in (Athanasiadis, Martin & Stratis, 2001 and 2003). Electromagnetic direct and low-frequency inverse scattering problems corresponding to the interior or exterior excitation of a layered sphere by a point dipole were investigated in (Tsitsas & Athanasiadis, 2006) and (Tsitsas, 2009).

It is worth to note that the scattering of spherical electromagnetic waves by piecewise homogeneous scatterers has interesting physical and technological applications. One of the most representative applications is the study of the interaction between the antenna of the mobile phone and the human head; the generated electromagnetic field is spherical due to the close proximity of the mobile phone antenna to the human head. The determination of the total radiation, which is absorbed by the head, provides important information related to the biological effects of electromagnetic radiation.

In this Thesis, analytical methodologies are investigated, treating the inverse scattering problem of a normalized spherical electromagnetic wave by a layered spherical scatterer. More precisely, a 2-layered spherical scatterer is considered where the first layer is dielectric and the second layer (core) is either dielectric or perfectly electric conducting. The inverse scattering problems under examination concern the determination of the layers' radii and the location of the sphere's center as well as the physical characteristics of the scatterer from known data of the scattered electromagnetic field.

The direct electromagnetic scattering problem is solved analytically in every region by imposing a combined Sommerfeld T-matrix algorithm. Then, the exact field expressions are approximated in the low-frequency realm in order to derive useful asymptotic expansions of the far-field pattern and the scattering cross section. These low-frequency expansions are effectively utilized in the establishment of the above mentioned inverse scattering algorithms. In particular, first, a geometric method is described for the solution of the inverse scattering problem referring to the determination of the sphere's center and the layers radii for known physical characteristics of the sphere. Then, the inverse problem of determining the material parameters of the 2-layered sphere for known geometrical characteristics is considered. The latter problem requires the use of interior as well as exterior primary dipole excitations.

II. METHODOLOGY

A. *Mathematical Formulation*

Consider a spherical scatterer V with radius a_1. The interior of V is divided into two spherical layers V_1 and V_2, respectively, defined by $a_2<r<a_1$ and $0\leq r<a_2$. Layer V_1 is occupied by a material with dielectric permittivity ε_1 and magnetic permeability μ_1. The core V_2 is either dielectric specified by physical constants ε_2 and μ_2 or perfect electric

conducting (PEC). The exterior V_0 ($r>a_1$) of the scatterer V is homogeneous with permittivity ε_0 and permeability μ_0.

The scatterer V is excited by a time-harmonic spherical electromagnetic wave, generated by a magnetic dipole at $\mathbf{r}_q$ of layer V_q (q=0,1,2) with moment $\hat{\mathbf{r}}_q \times \hat{\mathbf{p}}$. The primary electric field radiated by this dipole is expressed by

$$\mathbf{E}^{\mathrm{pr}}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = \frac{r_q e^{-ik_q r_q}}{ik_q} \nabla \times \left[\frac{e^{ik_q|\mathbf{r}-\mathbf{r}_q|}}{|\mathbf{r}-\mathbf{r}_q|} \hat{\mathbf{r}}_q \times \hat{\mathbf{p}} \right], \ \mathbf{r} \neq \mathbf{r}_q , \quad (1)$$

where r_q=|$\mathbf{r}_q$| and k_q denotes the wavenumber of layer V_q. For a dipole receding to infinity, this spherical wave reduces to a unit amplitude plane wave with direction of propagation $-\hat{\mathbf{r}}_q$ and polarization $\hat{\mathbf{p}}$ (Tsitsas, 2009).

The total electric field in V_q is defined as the superposition of the primary and the secondary field

$$\mathbf{E}^{q}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{E}^{\mathrm{pr}}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) + \mathbf{E}^{\mathrm{sec}}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}), \ \mathbf{r} \in V_q \setminus \{\mathbf{r}_q\} , \quad (2)$$

while the total field in V_j ($j\neq q$) is the secondary field $\mathbf{E}^{j}_{\mathbf{r}_q}$.

The primary and secondary fields in the above described spherical scatterer satisfy Maxwell's equations with linear isotropic constitutive relations (determined by scalar piecewise constant permittivity and permeability) and with vanishing electric charge and current density. By combining Maxwell's equations with the constitutive relations, it is verified that the electric field in layer V_j satisfies the vector Helmholtz equation

$$\nabla^2 \mathbf{E}^{j}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) + k_j^2 \mathbf{E}^{j}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{0} , \quad (3)$$

for $\mathbf{r} \in V_j$ if $j \neq q$ and $\mathbf{r} \in V_q \setminus \{\mathbf{r}_q\}$ if $j = q$.

The total electric fields satisfy the transmission boundary conditions on the spherical surfaces $r=a_j$ (j=1,2)

$$\left.\begin{aligned} &\hat{\mathbf{r}} \times \mathbf{E}^{j-1}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = \hat{\mathbf{r}} \times \mathbf{E}^{j}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) \\ &\hat{\mathbf{r}} \times \nabla \times \mathbf{E}^{j-1}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = (\mu_{j-1}/\mu_j) \hat{\mathbf{r}} \times \nabla \times \mathbf{E}^{j}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) \end{aligned}\right\} . \quad (4)$$

For a PEC core, conditions (4) hold only for j=1, while on $r=a_2$ the PEC boundary condition holds

$$\hat{\mathbf{r}} \times \mathbf{E}^{1}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{0} . \quad (5)$$

The total electric field in the unbounded domain V_0 satisfies the Silver-Müller radiation condition (Colton & Kress, 1992)

$$\lim_{r\to\infty} \left[\hat{\mathbf{r}} \times \nabla \times \mathbf{E}^{0}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) + ik_0 r \mathbf{E}^{0}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) \right] = \mathbf{0} , \quad (6)$$

uniformly for all directions $\hat{\mathbf{r}}$ in the unit sphere S^2.

The secondary and the total electric fields in V_0 have, respectively, the asymptotic expressions for $r\to\infty$,

$$\begin{aligned} &\mathbf{E}^{\mathrm{sec}}_{\mathbf{r}_0}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{g}_{\mathbf{r}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}) h_0(k_0 r) + O(r^{-2}), \\ &\mathbf{E}^{0}_{\mathbf{r}_q}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{g}_{\mathbf{r}_q}(\hat{\mathbf{r}};\hat{\mathbf{p}}) h_0(k_0 r) + O(r^{-2}), \quad q \geq 1, \end{aligned} . \quad (7)$$

where h_0 the zero-th order spherical Hankel function of the first kind, while the *q-excitation far-field pattern* $\mathbf{g}$ describes the response of the scatterer in the far-field, due to the excitation by the primary field in layer V_q.

The *q-excitation total cross-section* is defined by

$$\sigma_{\mathbf{r}_q} = \frac{1}{k_0^2} \int_{S^2} |\mathbf{g}_{\mathbf{r}_q}(\hat{\mathbf{r}};\hat{\mathbf{p}})|^2 \mathrm{d}s(\hat{\mathbf{r}}) , \quad (8)$$

and represents the average of the far-field's power radiated over all directions, due to the dipole excitation in layer V_q.

B. *Solution of the Direct Scattering Problem*

The solution of the direct scattering problem is obtained by imposing a combined Sommerfeld T-matrix analytic algorithm (Tsitsas, 2009). The Sommerfeld's method (Sommerfeld, 1949) handles the singularity of the dipole field, while the T-matrix method (Valagiannopoulos & Tsitsas, 2009) the effect of the sphere's layers. More precisely, the primary and secondary fields in every layer are expressed as series of the spherical vector wave functions. The unknown coefficients in the secondary fields' expansions are, subsequently, determined by applying a T-matrix method. The procedure is analyzed in (Tsitsas, 2009) while details of the calculations are included in (Katsimaglis, 2013).

The q-excitation far-field pattern, as determined by applying the above mentioned algorithmic procedure, is

$$\begin{aligned} &\mathbf{g}_{\mathbf{r}_q}(\hat{\mathbf{r}};\hat{\mathbf{x}}) = -\sum_{n=1}^{\infty} \frac{(2n+1)(-i)^n}{\sqrt{n(n+1)}} \times \\ &\left\{ a^0_{q,n} \frac{h_n(k_q r_q)}{h_0(k_q r_q)} \mathbf{B}_{e1n}(\hat{\mathbf{r}}) + i\beta^0_{q,n} \frac{\hat{h}'_n(k_q r_q)}{\hat{h}_0(k_q r_q)} \mathbf{C}_{o1n}(\hat{\mathbf{r}}) \right\} \end{aligned} , \quad (9)$$

where functions $\mathbf{B}$ and $\mathbf{C}$ are defined in (Morse & Feshbach, 1953), $a^0_{q,n}$ and $\beta^0_{q,n}$ are determined coefficients, while $\hat{h}_n(z) = z h_n(z)$. The q-excitation total cross-section is calculated by combining (8) and (9), yielding the final result

$$\sigma_{\mathbf{r}_q} = \frac{2\pi}{k_0^2} \sum_{n=1}^{\infty} (2n+1) \left[\left|\gamma_{q,n}\right|^2 + \left|\delta_{q,n}\right|^2 \right], \quad (10)$$

where $\gamma_{q,n}$ and $\delta_{q,n}$ are connected to $a^0_{q,n}$ and $\beta^0_{q,n}$.

C. *Low-Frequency Approximations*

The exact far-field solutions (9) and (10) are expressed as complicated series expansions. They can be simplified and become more workable by imposing the *low-frequency assumption* k_0a_1<<1, namely by assuming that the sphere's radius a_1 is much smaller than the primary field's wavelength. Under this assumption, the approximations of the q-excitation far-field pattern, for the case of a PEC core, are found to be

$$\begin{aligned} &\sigma_{\mathbf{r}_0} = \pi a_1^2 \{ \tfrac{2}{3} (Q_{0,1})^2 \tau_0^4 + \tfrac{2}{3} k_0^2 a_1^2 \tau_0^2 \times \\ &[4(P_{0,1})^2 - (Q_{0,1})^2 + \tfrac{4\tau_0^4}{15} (Q_{0,2})^2]\} + O((k_0 a_1)^4) \end{aligned} \quad (11)$$

and

$$\begin{aligned} &\sigma_{\mathbf{r}_1} = \pi a_1^2 \{ \tfrac{6}{\eta_1^4} (Q_{1,1})^2 \tau_1^4 + \tfrac{6}{\eta_1^2} k_0^2 a_1^2 \tau_1^2 \times \\ &[(P_{1,1})^2 - (Q_{1,1})^2 + \tfrac{5\tau_1^4}{27\eta_1^4} (Q_{1,2})^2]\} + O((k_0 a_1)^4) \end{aligned} , \quad (12)$$

where P and Q are determined coefficients, while $\tau_i=a_1/r_i$ (i=1,2), and $\eta_1=k_1/k_0$.

Similar low-frequency approximations can be derived for the case of a dielectric core.

The obtained simplified low-frequency expressions can be effectively utilized in the establishment of inverse scattering algorithms concerning the determination of the scattering problem's parameters via combinations of certain far-field measurements.

III. Selected Results

In this section, certain selected results are included concerning developed low-frequency inverse scattering algorithms for the determination of the geometrical and physical characteristics of the 2-layered sphere.

A. *Localization of the Sphere and Reconstruction of its Geometrical Characteristics*

We seek to determine the center's coordinates and the layers radii of a 2-layered sphere with a PEC core and given dielectric permittivity and magnetic permeability of the covering layer. We consider the Cartesian coordinate system *Oxyz* and for a chosen fixed length ℓ the five dipole locations (0,0,0), (ℓ,0,0), (0,ℓ,0), (0,0,ℓ), and (0,0,2ℓ) with unknown distances b_1, b_2, b_3, b_4, and b_5 from the sphere's center (see Fig. 1). For each dipole's location we measure the leading-order low-frequency term of the 0-excitation total cross-section (11), and obtain the five measurements

$$m_j = \tfrac{2}{3}\pi a_1^2 (Q_{0,1}(a_1,a_2))^2 (a_1/b_j)^4, \; j=1,...,5 . \tag{13}$$

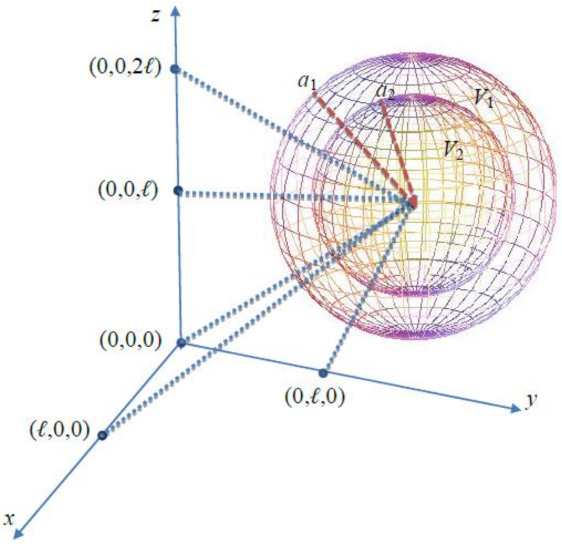

Figure 1. The five dipoles' locations utilized for the formulation of the inverse scattering algorithm determining the center's coordinates and the layers radii of a 2-layered sphere with a PEC core.

Next, we use the dimensionless normalization

$$\gamma_j = \frac{\ell}{\sqrt{m_j}} = \sqrt{\tfrac{3}{2\pi}}\,\frac{\ell}{a_1 Q_{0,1}(a_1,a_2)}\left(\frac{b_j}{a_1}\right)^2 . \tag{14}$$

Eqs. (14) constitute a system of five equations with the seven unknowns a_1, a_2, and b_j. The sixth equation is the law of cosines

$$b_5^2 = 2\ell^2 + 2b_4^2 - b_1^2 , \tag{15}$$

which by means of (14) is written in normalized form as

$$\gamma_5 = \sqrt{\tfrac{6}{\pi}}\,\frac{\ell^3}{a_1^3 Q_{0,1}(a_1,a_2)} + 2\gamma_4 - \gamma_1 . \tag{16}$$

The seventh equation is derived by measuring the cross-section for a dipole far away from the sphere (so that we admit plane wave incidence). This measurement is obtained from (11) for $\tau_0 \to 0$ (see also (Tsitsas, 2009)), yielding

$$m_6 = \tfrac{2}{3}\pi a_1^2 (k_0 a_1)^4 [4(P_{0,1}(a_1,a_2))^2 + (Q_{0,1}(a_1,a_2))^2] \tag{17}$$

By eliminating the factor $Q_{0,1}a_1^3$ between (14) and (16), we get

$$\left(\frac{b_j}{\ell}\right)^2 = \frac{2\gamma_j}{\gamma_5 - 2\gamma_4 + \gamma_1} , \tag{18}$$

which determines the distances b_j. Then, the center of the 2-layered sphere coincides with the intersection point of the four spheres centered at (0,0,0), (ℓ,0,0), (0,ℓ,0), (0,0,ℓ) with determined radii b_1, b_2, b_3, b_4.

The layers radii a_1 and a_2 are determined by the solution of the following 2×2 non-linear system of (14) for j=1 and (17)

$$a_1^3 Q_{0,1}(a_1,a_2) = \sqrt{\tfrac{3}{2\pi}}\,\ell b_1^2/\gamma_1 ,$$
$$a_1^3 P_{0,1}(a_1,a_2) = \sqrt{\tfrac{3}{8\pi}}\sqrt{\frac{m_6}{k_0^4} - \frac{\ell^2 b_1^4}{\gamma_1^2}} . \tag{19}$$

B. *Reconstruction of the Sphere's Physical Characteristics*

Now, we seek to determine the relative dielectric permittivities and magnetic permeabilities of the sphere's layers for given center's coordinates and layers radii. The inverse scattering algorithm is developed for a 2-layered dielectric sphere. Hence, we will determine the four material parameters ε_{r1}, ε_{r2}, μ_{r1}, and μ_{r2} for given radii a_1 and a_2.

Consider an exterior dipole at (0,0,b_0) with known distance $b_0 > a_1$ from the sphere's center (see Fig. 2) and measure the leading order m_1 and second-order m_2 low-frequency terms of the corresponding 0-excitation total cross-section's expansion (see (4.17) of (Tsitsas, 2009)). Thus, we obtain

$$m_1 = \tfrac{8}{3}\pi \frac{a_1^6}{b_0^4}(S_{0,1}(\mu_{r1},\mu_{r2}))^2 , \tag{20}$$

and

$$m_2 = \tfrac{8}{3}\pi k_0^2 \frac{a_1^6}{b_0^4}\Big[(R_{0,1}(\varepsilon_{r1},\varepsilon_{r2}))^2 - (S_{0,1}(\mu_{r1},\mu_{r2}))^2 + \tfrac{3}{5}\frac{a_1^4}{b_0^4}(S_{0,2}(\mu_{r1},\mu_{r2}))^2\Big] , \tag{21}$$

where the R and the S parameters are defined in (Tsitsas, 2009).

Next, consider an interior dipole at (0,0,b_1) with known distance $a_2 < b_1 < a_1$ from the center (see Fig. 2) and measure the leading order term m_3 of the 1-excitation total cross-section's expansion (see (4.18) of (Tsitsas, 2009)). This, yields

$$m_3 = 24\pi \frac{a_1^6}{b_1^4} \frac{(S_{1,1}(\varepsilon_{r1}, \mu_{r1}, \mu_{r2}))^2}{\eta_1^4}. \qquad (22)$$

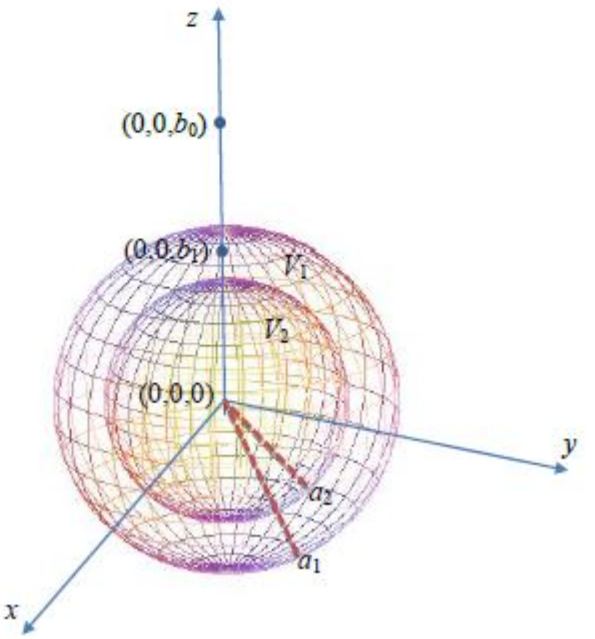

Figure 2. The two dipoles' locations utilized for the formulation of the inverse scattering algorithm determining the material parameters of a 2-layered sphere with a dielectric core.

Finally, measure the leading-order low-frequency term m_4 for plane wave incidence (by removing the exterior dipole far away from the sphere) to get

$$m_4 = \tfrac{8}{3}\pi k_0^4 a_1^6 \left[(R_{0,1}(\varepsilon_{r1}, \varepsilon_{r2}))^2 + (S_{0,1}(\mu_{r1}, \mu_{r2}))^2 \right]. \qquad (23)$$

Now, Eqs. (20) and (22) take the form

$$(S_{0,1}(\mu_{r1}, \mu_{r2}))^2 = \tfrac{3}{8\pi} \frac{b_0^4}{a_1^6} m_1, \qquad (24)$$

and

$$\frac{(S_{1,1}(\varepsilon_{r1}, \mu_{r1}, \mu_{r2}))^2}{\eta_1^4} = \tfrac{1}{24\pi} \frac{b_1^4}{a_1^6} m_3. \qquad (25)$$

Besides, by combining (20) with (23), we have

$$(R_{0,1}(\varepsilon_{r1}, \varepsilon_{r2}))^2 = \tfrac{3}{8\pi} \frac{1}{k_0^4 a_1^6} \left(m_4 - k_0^4 b_0^4 m_1 \right), \quad (26)$$

while by combining (20) and (21) with (26), we obtain

$$(S_{0,1}(\mu_{r1}, \mu_{r2}))^2 = \tfrac{5}{8\pi} \frac{b_0^6}{k_0^2 a_1^{10}} \left(m_2 - \frac{m_4}{k_0^2 b_0^2} + 2k_0^2 b_0^2 m_1 \right). \qquad (27)$$

Eqs. (24)-(27) constitute a 4×4 non-linear system with respect to the unknown physical parameters ε_{r1}, μ_{r1}, and ε_{r2}, μ_{r2} of layers V_1 and V_2, respectively.

The inverse scattering problem for a 2-layered sphere with PEC core can be handled by a similar and simpler procedure. For this type of problem we need to determine the two unknown material parameters ε_{r1} and μ_{r1} of layer V_1 for given radii a_1 and a_2. Thus, two measurements will be sufficient.

IV. CONCLUSIONS

In this Thesis, simple and efficient algorithms concerning the inverse low-frequency electromagnetic scattering problem of a piecewise homogeneous spherical scatterer were investigated. The primary field was due to a point dipole lying in the interior or the exterior of the spherical scatterer. Two particular far-field inverse problems were examined: the first concerned the determination of the layers' radii and the location of the sphere's center and the second the determination of the physical characteristics of the sphere.

Interesting future research directions concern the investigations of the corresponding direct and inverse scattering problems with point dipoles lying at arbitrary locations inside or outside the sphere (not necessarily on the *z*-axis) and possessing arbitrary polarizations.

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